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Annihilating polynomials of excellent quadratic forms

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Abstract. If φ is an excellent form, then it is possible to use the dimensions of the higher complements of φ to obtain an annihilating polynomial of φ of low degree. The main result of this paper is the construction of such a polynomial with the help of methods from the theory of generic splitting of quadratic forms.

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1. Introduction. In [5] Lewis showed that for each $n \in \mathbb{N}_0$ the polynomial

$$\mathbb{Z}[X] \ni P_n := \begin{cases} X(X^2-2)\cdots(X^2-(n-2)^2)(X^2-n^2) & \text{for } n \text{ even} \\ (X^2-1)(X^2-3^2)\cdots(X^2-(n-2)^2)(X^2-n^2) & \text{for } n \text{ odd} \end{cases}$$

annihilates every n-dimensional quadratic form over an arbitrary field K. Since then a number of classes of quadratic forms have been identified for which annihilating polynomials of significantly lower degree can be constructed, e.g. positive forms and trace forms. An extensive survey of the currently known results about annihilating polynomials of quadratic forms can be found in [6].

In this paper we will consider the class of excellent forms. Let φ be an excellent form of dimension n over K, let h be its height, and for $j = 0, \dots, h$ let n_j be the dimension of the j-th complement of φ. We will show that

$$\mathbb{Z}[X] \ni E_n := \begin{cases} X(X^2-n_{h-1}^2)\cdots(X^2-n_1^2)(X^2-n^2) & \text{for } n \text{ even} \\ (X^2-1)(X^2-n_{h-1}^2)\cdots(X^2-n_1^2)(X^2-n^2) & \text{for } n \text{ odd} \end{cases}$$

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annihilates φ. If φ is isotropic, then we can prove this just by using the fact that φ is a Pfister neighbour of a hyperbolic Pfister form. If φ is anisotropic we will use methods from the theory of generic splitting of quadratic forms.

2. Notation and Basic Facts. The reader is expected to be familiar with the algebraic theory of quadratic forms and the theory of generic splitting of quadratic forms. An introduction to quadratic forms can be found in [7], [8] or [4]. The theory of generic splitting of quadratic forms was introduced in [2].

We define the set of natural numbers $\mathbb{N}$ not to contain 0. If we need 0 as a natural number we use the notation $\mathbb{N}_0$. In all of this paper K will be a field with $\mathrm{char}(K) \neq 2$, and we only consider non-degenerate quadratic forms over K. Often we will simply use the notion "form" when we mean a quadratic form.

Let φ be a quadratic form over K. We denote by $\dim(\varphi)$ the dimension of the quadratic space over K corresponding to φ. If ψ is another quadratic form over K, then we write $\varphi \perp \psi$ and $\varphi \otimes \psi$ for the *orthogonal sum* and the *tensor product* of φ and ψ. In case there exists a form χ over K such that $\varphi \cong \psi \perp \chi$ we call ψ a *subform* of φ. For $n \in \mathbb{N}_0$ the n-fold sum of φ will be denoted by $n \times \varphi$. If $\varphi = \langle a_1, \dots, a_n \rangle$ with $a_1, \dots, a_n \in K^*$, then $a\varphi := \langle aa_1, \dots, aa_n \rangle$ for $a \in K^*$. In particular, if $a = -1$, we write $-\varphi := \langle -a_1, \dots, -a_n \rangle$. For $a_1, \dots, a_k \in K^*$ we use the notation $\langle\langle a_1, \dots, a_k \rangle\rangle$ for the *k-fold Pfister form* $\bigotimes_{i=1}^{k} \langle 1, a_i \rangle$. If there exists some $a \in K^*$ such that $\varphi \cong a\psi$ we say that φ and ψ are *similar*. We denote by $\widehat{W}(K)$, respectively $W(K)$, the *Witt-Grothendieck ring*, respectively *Witt ring*, of K. We write $[\varphi]$ for the *isometry class* of φ in $\widehat{W}(K)$, and if φ is isometric to ψ we use the notation $\varphi \cong \psi$. In the case that $\varphi \cong n \times \langle 1 \rangle$ we simply write n instead of $[n \times \langle 1 \rangle]$ for the isometry class of φ in $\widehat{W}(K)$. The sum respectively product of $[\varphi]$ and $[\psi]$ in $\widehat{W}(K)$ will be denoted by $[\varphi] + [\psi]$ respectively $[\varphi]\,[\psi]$ (or $[\varphi] \cdot [\psi]$). For an arbitrary element $[\varphi] - [\psi] \in \widehat{W}(K)$ we can extend the notion of dimension by setting $\dim([\varphi] - [\psi]) := \dim(\varphi) - \dim(\psi)$. If the forms φ and ψ represent the same element in $W(K)$ they are called *equivalent*, and we write $\varphi \sim \psi$. For an element $a \in K^*$ its class in the *square class group* $K^*/(K^*)^2$ will be denoted by $\overline{a} := a(K^*)^2$.

If L is a field extension over K, then we write φ_L for the form φ considered as a quadratic form over L. Most important in this paper is the case that L is the function field $K(\varphi)$ of φ. To be exact, if $\dim(\varphi) \geq 2$ and $\varphi \not\cong \langle 1, -1 \rangle$ we denote by $K(\varphi)$ the function field of the quadric defined by φ, otherwise we set $K(\varphi) := K$. Now consider a generic splitting tower $K_0 := K \subset K_1 \subset \cdots \subset K_h$ of φ (see [3, Section 5]), then h is called the *height* of φ. For $j = 0, \dots, h$ we set $\varphi_j := (\varphi_{K_j})_{\mathrm{an}}$, where $(\varphi_{K_j})_{\mathrm{an}}$ denotes the anisotropic kernel of φ_{K_j}, and call φ_j the *j-th anisotropic kernel* of φ. The form φ is a *Pfister neighbour* if there exists a Pfister form τ over K such that φ is similar to a subform of τ and $\dim(\varphi) > \frac{1}{2}\dim(\tau)$. If χ is a form over K and $a \in K^*$ such that $a\tau \cong \varphi \perp \chi$, then χ is called the *complement* of φ. We say that φ is *excellent* if there exists a sequence

$\psi_0 := \varphi, \psi_1, \ldots, \psi_h$ of forms over K such that $\dim(\psi_h) \leq 1$ and ψ_{i-1} is a Pfister neighbour with complement ψ_i for $i = 1, \ldots, h$. The ψ_i are the *higher complements* of φ. In this setting, if φ is anisotropic, it is well-known that $(\psi_j)_{K_j} \cong (-1)^j \varphi_j$ for $j = 0, \ldots, h$ (see [3, Proposition 7.9]).

Definition 2.1. Let R be a unitary, commutative ring, and let $\iota : \mathbb{Z} \to S$ be the canonical homomorphism defined by $\iota(1) = 1_S$. If $P = a_n X^n + \ldots a_1 X + a_0 \in \mathbb{Z}[X]$ is a polynomial and $x \in S$, then P is called an *annihilating polynomial of* x if

$$P(x) \;:=\; \iota(a_n)x^n + \cdots + \iota(a_1)x + \iota(a_0) \;=\; 0.$$

Usually we will omit the ι and just write $as := \iota(a)s$ for $a \in \mathbb{Z}$ and $s \in S$.

Let $G := {}^{K^*}/(K^*)^2$ be the square class group of K. We use Hurrelbrink's approach in [1] to construct annihilating polynomials for elements of the group ring $\mathbb{Z}[G]$:

Consider the *dual group*

$$\widehat{G} \;:=\; \left\{\chi : G \to S^1 \;\middle|\; \chi \text{ is a group homomorphism}\right\}$$

of G, where $S^1 \subset \mathbb{C}$ is the unit circle. Then each *character* $\chi \in \widehat{G}$ can be extended to a $\mathbb{Z}$-algebra homomorphism

$$\underline{\chi} : \quad \mathbb{Z}[G] \longrightarrow \mathbb{C}, \quad \sum_{g \in G} z_g g \longmapsto \sum_{g \in G} z_g \chi(g).$$

For $f \in \mathbb{Z}[G]$ the set $S_f := \left\{\underline{\chi}(f) \,\middle|\, \chi \in \widehat{G}\right\}$ is finite by [1, Observation 1.2], and it is easy to see that it is a subset of $\mathbb{Z}$. This allows us to define the polynomial

$$P_f \;:=\; \prod_{\underline{\chi}(f) \in S_f} \big(X - \underline{\chi}(f)\big) \;\in\; \mathbb{Z}[X],$$

which by [1, Theorem 1.3] is an annihilating polynomial of f.

Now consider the canonical surjection $\pi_1 : \mathbb{Z}[G] \to \widehat{W}(K)$ defined by

$$\overline{a_1} + \cdots + \overline{a_n} \longmapsto \langle a_1, \ldots, a_n \rangle.$$

Definition 2.2. An element $f = \sum_{g \in G} z_g g \in \mathbb{Z}[G]$ is called a *preform* if $z_g \geq 0$ for all $g \in G$. The integer $n := \sum_{g \in G} z_g$ is the *dimension* of f, and we write $\dim(f) := n$. If $f = \overline{a_1} + \cdots + \overline{a_n}$, and if φ is a quadratic form over K, then f is called a *preform of* φ if we have $\varphi \cong \langle a_1, \ldots, a_n \rangle$ or equivalently $\pi_1(f) = [\varphi]$.

Definition 2.3. Let φ be a quadratic form over K. A polynomial $P \in \mathbb{Z}[X]$ is called an *annihilating polynomial of* φ if it is an annihilating polynomial of $[\varphi] \in \widehat{W}(K)$.

Corollary 2.4. *If $f \in \mathbb{Z}[G]$ is a preform of a quadratic form φ over K, then P_f is an annihilating polynomial of φ.*

Proof. We have

$$P_f\big([\varphi]\big) \;=\; P_f\big(\pi_1(f)\big) \;=\; \pi_1\big(P_f(f)\big) \;=\; \pi_1(0) \;=\; 0.$$

□

Remark 2.5. If $f \in \mathbb{Z}[G]$ is a preform of an arbitrary form φ over K, $\dim(\varphi) = n$, then it is easy to see that $S_f \subset \{-n, -n+2, \ldots, n-2, n\}$ (see [1, Observation 2.2]). We thus obtain the result first presented by Lewis in [5], that the polynomial

$$P_n \;:=\; (X+n)(X+n-2)\cdots(X-n+2)(X-n) \;\in\; \mathbb{Z}[X]$$

annihilates all n-dimensional quadratic forms over K.

3. Main Result.

Lemma 3.1. *If φ and ψ are quadratic forms over K with $\varphi \sim \psi$, and if $P \in \mathbb{Z}[X]$ is an annihilating polynomial of φ, then $P([\psi]) \sim 0$.*

Proof. Let $\pi_2 : \widehat{W}(K) \to W(K)$ be the natural projection. Then

$$0 \;=\; \pi_2(P([\varphi])) \;=\; P(\pi_2([\varphi])) \;=\; P(\pi_2([\psi])) \;=\; \pi_2(P([\psi]))$$

or equivalently $P([\psi]) \sim 0$. □

Proposition 3.2. *Let φ be a Pfister neighbour of dimension $n > 0$ with complement $-\psi$ over K, and let $a \in K^*$ and τ be a Pfister form such that $\varphi \perp -\psi \cong a\tau$. If $Q \in \mathbb{Z}[X]$ is an annihilating polynomial of ψ, then $P := Q \cdot (X^2 - n^2)$ is an annihilating polynomial of φ. In the case that $a \in (K^*)^2$ (in particular if φ is isotropic) the polynomial $\widetilde{P} := Q \cdot (X - n)$ suffices.*

Proof. If φ is isotropic, then τ is isotropic as well and therefore hyperbolic. Hence $\varphi \sim \psi$. By the previous lemma $Q([\varphi]) \sim 0$, and therefore

$$Q([\varphi]) \cdot [\varphi] \;\sim\; 0 \;\sim\; Q([\varphi]) \cdot n.$$

The claim follows since $\dim(Q([\varphi]) \cdot [\varphi]) = \dim(Q([\varphi]) \cdot n)$.

Now let φ be anisotropic. Since $\varphi_{K(\varphi)} \sim \psi_{K(\varphi)}$, we obtain $Q([\varphi_{K(\varphi)}]) \sim 0$ from the previous lemma. A result by Knebusch ([2, Example 4.1]) states that $Q([\varphi_{K(\tau)}]) \sim 0$. Hence by [2, Lemma 4.4] there exists an element $x \in \widehat{W}(K)$ such that $Q([\varphi]) = x \cdot [\tau]$. Now $a\varphi \subset \tau$. Since $b\tau \cong \tau$ for all $b \in K^*$ represented by τ, it follows that $a\varphi \otimes \tau \cong n \times \tau$. If $a \in (K^*)^2$, then

$$Q([\varphi]) \cdot ([\varphi] - n) \;=\; x \cdot [\tau] \cdot ([\varphi] - n) \;=\; x \cdot ([\varphi \otimes \tau] - [n \times \tau]) \;=\; x \cdot 0 \;=\; 0.$$

Otherwise, if a is not a square, we still obtain

$$\begin{aligned} Q([\varphi]) \cdot \Big([\varphi]^2 - n^2\Big) \;&=\; x \cdot [\tau] \cdot \Big([\varphi]^2 - n^2\Big) \;=\; x \cdot \Big([\tau] \cdot [a\varphi]^2 - [\tau] \cdot n^2\Big) \\ &=\; x \cdot \big([a\varphi \otimes a\varphi \otimes \tau] - \big[n^2 \times \tau\big]\big) \;=\; x \cdot 0 \;=\; 0. \end{aligned}$$

□

Theorem 3.3. *Let φ be an excellent form of height $h \in \mathbb{N}_0$ over K, let $\varphi = \psi_0, \psi_1, \ldots, \psi_h$ be the sequence of its higher complements, and for $j = 0, \ldots, h$ set $n_j := \dim(\psi_j)$. Then*

$$\mathbb{Z}[X] \ni E_n := \begin{cases} X(X^2 - n_{h-1}^2) \cdots (X^2 - n_1^2)(X^2 - n^2) & \text{for } n \text{ even} \\ (X^2-1)(X^2 - n_{h-1}^2) \cdots (X^2 - n_1^2)(X^2 - n^2) & \text{for } n \text{ odd} \end{cases}$$

annihilates φ.

Proof. We proceed by induction on h. If $h = 0$ and $\dim(\varphi) = 0$, then indeed $E_0 = X$ annihilates $\varphi = 0$. In the case $\dim(\varphi) = 1$ we have $\varphi = \langle a \rangle$ for some $a \in K^*$, and $[a]^2 = 1$. This shows, that $E_1 = X^2 - 1$ annihilates φ.

Now let $h > 0$. By induction E_{n_1} annihilates the excellent form ψ_1. Since ψ_1 is the complement of φ it follows from the previous lemma that $E_{n_1} \cdot (X^2 - n^2) = E_n$ annihilates φ. □

Remark 3.4. It is well-known that the dimensions of the higher complements and the height of an excellent form φ only depend on the dimension of φ and can be explicitly calculated with the help of certain recursive functions (see [3, Corollary 7.11]).

Remark 3.5. Let φ be similar to a k-fold Pfister form, $k \in \mathbb{N}_0$. A consequence of proposition 3.2 is the well-known result that φ can be annihilated by $X(X^2 - 2^{2k}) = X(X - 2^k)(X + 2^k) \in \mathbb{Z}[X]$. If φ is a k-fold Pfister form it suffices to take the polynomial $X(X - 2^k)$.

There exists an easy proof of this result that only uses corollary 2.4: Let φ be similar to a k-fold Pfister form. Then there exist $g_0, g_1, \ldots, g_k \in G := K^*/(K^*)^2$ such that $f = g_0 \cdot (1 + g_1) \cdots (1 + g_k) \in \mathbb{Z}[G]$ is a preform of φ. Since for every $\chi \in \widehat{G}$ the map $\underline{\chi}$ is a homomorphism of rings, and since $\underline{\chi}(1 + g_i) \in \{0, 2\}$ for all $i = 1, \ldots, k$, it follows that $\underline{\chi}(f) \in \{-2^k, 0, 2^k\}$ and therefore $S_f \subset \{-2^k, 0, 2^k\}$. If $g_0 = 1$ we even get $S_f \subset \{0, 2^k\}$.

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